Combinatorial Models for Key and Atom Polynomials

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The Ring of Polynomials

$$\mathbb{Z}[x_1, x_2, \ldots, x_n] = \left\{ \begin{array}{c} \text{polynomials} \\ \text{in the variables } x_1, x_2, \ldots, x_n \\ \text{with integer coefficients} \end{array} \right\}.$$

Example: $3x_1^2 - 2x_2 + 5x_1x_2 \ \in \ \mathbb{Z}[x_1, x_2]$.

The Ring of Symmetric Polynomials

A polynomial is *symmetric* if it remains the same after permuting its variables.

Example: $x_1^2 + x_2^2 + x_3^2 \in \mathrm{Sym}_3$.

$$\mathrm{Sym}_n = \left\{ \begin{array}{c} \text{symmetric polynomials} \\ \text{in the variables } x_1, x_2, \ldots, x_n \\ \text{with integer coefficients} \end{array} \right\}$$

Rings are Modules

From any ring:

addition: $p + q$ multiplication: $p \cdot q$

...we can form a module by "forgetting" multiplication:

addition: $p + q$ scaling: np, $n \in \mathbb{Z}$

...which are like "vector spaces" but over a ring; for us, $\mathbb{Z}$.

Integer Basis

A subset B of a module M over $\mathbb{Z}$ is a basis if
for all $p \in M$, there exist unique finite linear combination:

$$p = \sum_{b \in B} c_b \cdot b,$$

where $c_b \in \mathbb{Z}$.

Monomial Basis of the Polynomial Ring

The set of all monomials forms a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$:

$$\left\{ x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \;\middle|\; \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0} \right\}$$

Example: $x_1 x_3^2 = x_1^1 x_2^0 x_3^2$ is a monomial in $\mathbb{Z}[x_1, x_2, x_3]$.

Each monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is defined by the sequence of its exponents.

Compositions index Monomials

A *composition* of length n is a sequence of nonnegative integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}{}^n.$$

Example: $(1, 0, 2)$ is a composition of length 3.

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Notation: $$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Example: $$x^{(1,0,2)} = x_1 x_3^2.$$

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The set of all monomials forms a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$:

$$\left\{ x^\alpha \mid \alpha \text{ is a composition of length } n \right\}.$$

Symmetric Monomial Basis of Sym_n

A *symmetric monomial* is the sum of all monomials obtained by rearranging the exponents of a monomial.

Example 1: $x_1^9x_2^7x_3^4 + x_1^9x_2^4x_3^7 + x_1^4x_2^9x_3^7 + x_1^7x_2^9x_3^4 + x_1^7x_2^4x_3^9 + x_1^4x_2^7x_3^9$.

Example 2: $x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4$.

Example 3: $x_1x_2x_3$.

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Example 2: $x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4$.

Example 3: $x_1x_2x_3$.

Each symmetric monomial is defined by the sequence of its exponents in decreasing order. In the examples:

$$(9,7,4), \quad (4,1,1), \quad (1,1,1)$$

Symmetric Monomial Basis of Sym_n

A *partition* of length n is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in {\mathbb{Z}_{\geq 0}}^n$$
$$\text{with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

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$$m_\lambda = \sum_{\text{rearrangements } \alpha \text{ of } \lambda} x^\alpha.$$

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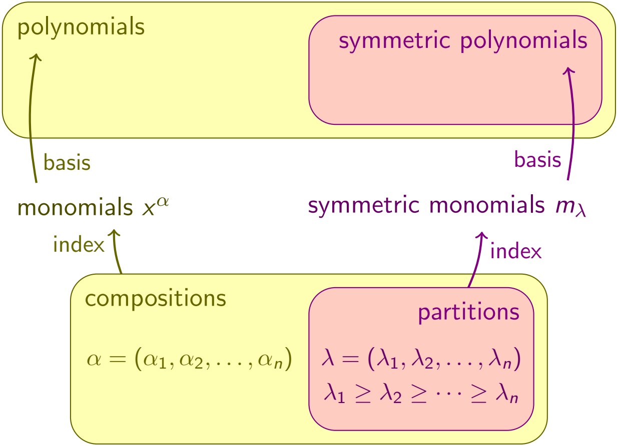
$$m_\lambda = \sum_{\text{rearrangements } \alpha \text{ of } \lambda} x^\alpha.$$

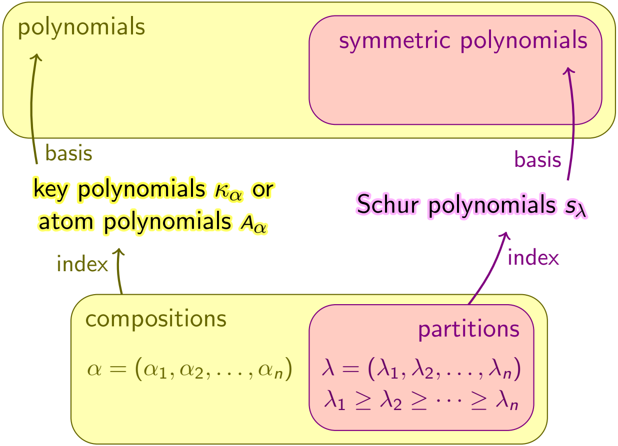
Sym_n has a basis of symmetric monomials:

$$\{ m_\lambda \mid \lambda \text{ is a partition of length } n \}.$$

Checkpoint


polynomials
symmetric polynomials
basis
basis
monomials x^α
symmetric monomials m_λ
index
index
compositions
partitions
$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$
$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$
$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

New Perspective



Big Picture:

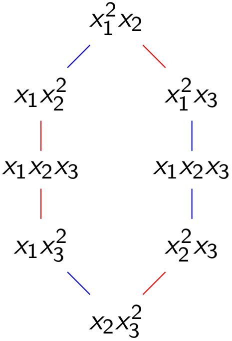
understanding κ_α, A_α $\Longrightarrow$ understanding $\mathbb{Z}[x_1, \ldots, x_n]$

understanding s_λ $\Longrightarrow$ understanding Sym_n

Some intuition on κ_α, A_α, and s_λ

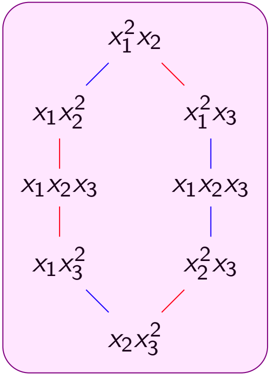
This diagram for partition $\lambda = (2, 1, 0)$ and its rearrangements

$$\alpha \quad = \quad (2,1,0), \quad (1,2,0), \quad (2,0,1), \quad (1,0,2), \quad (0,2,1), \quad (0,1,2)$$

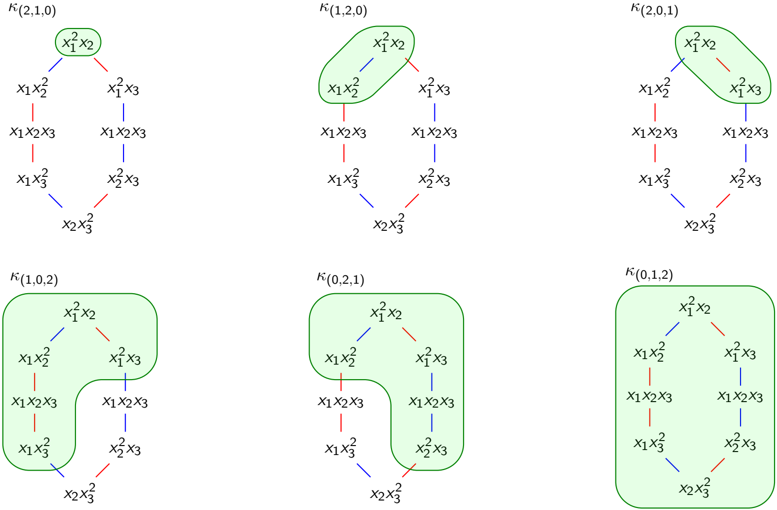


Example of s_λ

$$s_{(2,1,0)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

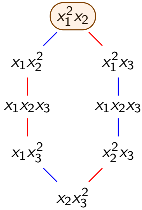


Example of κ_α

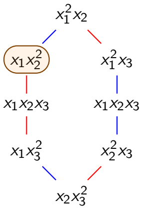


Example of A_α

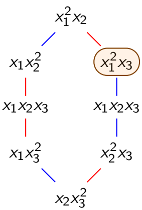
$A_{(2,1,0)}$



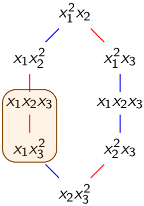
$A_{(1,2,0)}$



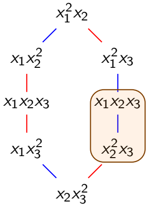
$A_{(2,0,1)}$



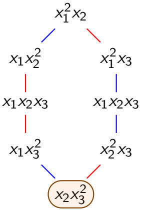
$A_{(1,0,2)}$



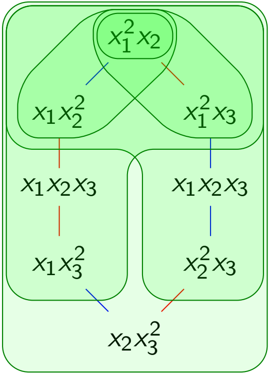
$A_{(0,2,1)}$

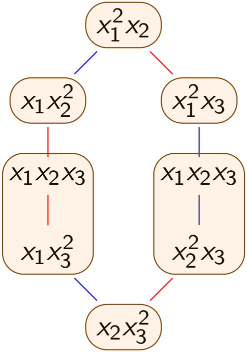


$A_{(0,1,2)}$



Comparing κ_α and A_α





Littlewood-Richardson Rule

The product of two Schur polynomials is a linear combination of Schur polynomials:

$$s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda,\mu}^\nu \cdot s_\nu, \qquad c_{\lambda,\mu}^\nu \in \mathbb{Z}.$$

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The Littlewood-Richardson Rule states that

$$c^\nu_{\lambda,\mu} = \begin{array}{c}\text{number of semistandard skew tableaux} \\ \text{of shape } \nu/\lambda \text{ and weight } \mu\end{array}$$

Corollary: $c^\nu_{\lambda,\mu}$ are nonnegative integers.

Product of Key Polynomials in Key Basis

The $\kappa_\alpha \cdot \kappa_\beta$ is a polynomial.
Thus, $\kappa_\alpha \cdot \kappa_\beta$ is a linear combination of key polynomials:

$$\kappa_\alpha \cdot \kappa_\beta = \sum_\gamma c_{\alpha,\beta}^{\gamma} \cdot \kappa_\gamma, \qquad c_{\alpha,\beta}^{\gamma} \in \mathbb{Z}.$$

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Research Question: Find a combinatorial description of the integer coefficients $c_{\alpha,\beta}^\gamma$ above.

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Spoiler: The coefficient $c_{\alpha,\beta}^{\gamma}$ are can be negative.

Example: $\kappa_{(0,1)}\kappa_{(1,0,1)} = \kappa_{(1,1,1)} + \kappa_{(1,2)} + \kappa_{(2,0,1)} - \kappa_{(2,1)}$.

Product of Key Polynomials in Atom Basis

$\kappa_\alpha \cdot \kappa_\beta$ is a linear combination of atom polynomials:

$$\kappa_\alpha \cdot \kappa_\beta = \sum_\gamma d_{\alpha,\beta}^\gamma \cdot A_\gamma, \qquad d_{\alpha,\beta}^\gamma \in \mathbb{Z}.$$

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Research Question: Find a combinatorial description of the integer coefficients $d^\gamma_{\alpha,\beta}$.

Conjecture (Reiner & Shimozono): The coefficients $d^\gamma_{\alpha,\beta}$ are nonnegative integers.

Many equivalent definitions

There are many equivalent definitions of key, atom, and Schur polynomials.

- using divided difference operators (more algebraic approach),
- using keys of Young tableaux (more combinatorial approach),
- using skyline augmented tableaux (another combinatorial approach),
- using Demazure crystals and Kashiwara operators (algebraic and combinatorial approach),
- many other equivalent definitions.

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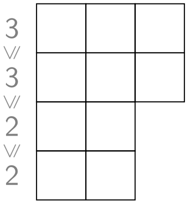
Young Diagram

A Young diagram is a collection of boxes arranged in left-justified rows and top-justified columns.

$$\text{partitions} \longleftrightarrow \text{Young diagrams}$$

"How many boxes are in the i-th row?"

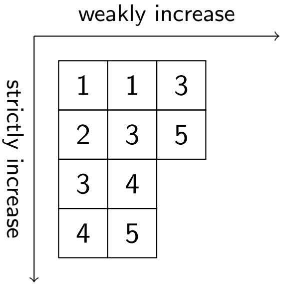
Example: $\lambda = (3, 3, 2, 2)$.



Semistandard Young Tableau (SSYT)

A SSYT is a filling of a Young diagram
with $\{1, 2, \ldots, n\}$ such that
the entries are weakly increasing along rows
and strictly increasing down columns.

Example: An SSYT of shape $\lambda = (3, 3, 2, 2)$.



Extracting monomials from SSYT

$$\begin{array}{rcl} \text{SSYTs} & \longrightarrow & \text{monomials} \\ T & \longmapsto & x^T \end{array}$$

Example:

1	1	3
2	3	5
3	4	
4	5	

$$\longmapsto \quad {x_1}^2 {x_2}^1 {x_3}^3 {x_4}^2 {x_5}^2$$

Schur Polynomial

For each partition λ, there is a Schur polynomial s_λ.

The Schur polynomial s_λ is the sum of all monomials corresponding to SSYTs of shape λ.

$$s_\lambda = \sum_{\text{SSYT } T \text{ of shape } \lambda} x^T.$$

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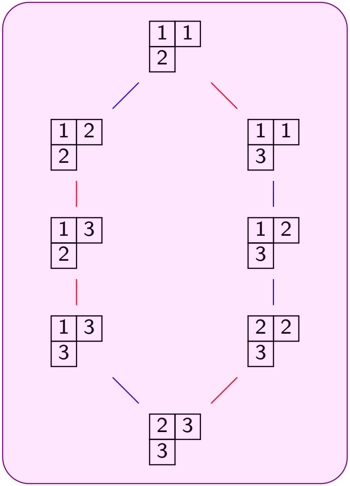
Fun Fact: The Schur polynomial s_λ is symmetric.

Example of Schur Polynomial

$$s_{(2,1,0)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$



Key

$$\begin{array}{rcl} \text{compositions} & \longrightarrow & \text{SSYTs} \\ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) & \longmapsto & \text{key}\,\alpha \end{array}$$

Example:

$$(1, 3, 0, 4, 2) \longmapsto$$

1	2	2	4
2	4	4	
4	5		
5			

Attention: Not all SSYTs can be obtained as keys.

Right Key of a SSYT

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There is a process to obtain the right key of a SSYT,
by making the entries slightly larger (not defined here).

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Example

$T =$

1	1	1	3	5
2	2	3	4	6
4	4	6		
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$K_+(T) =$

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3				
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6				

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5	6	6		
6				

Atom Polynomial

The atom polynomial A_α is the sum of all monomials corresponding to SSYTs whose right key is key(α).

$$A_\alpha = \sum_{\substack{\text{SSYT } T \\ K_+(T) = \text{key}(\alpha)}} x^T.$$

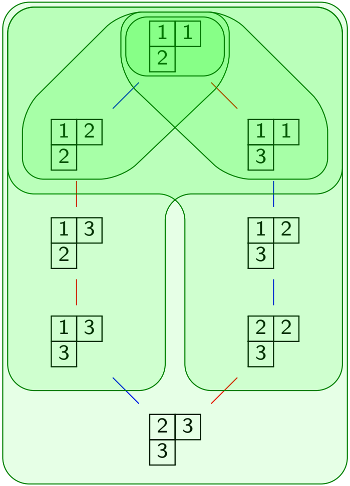
Key Polynomial

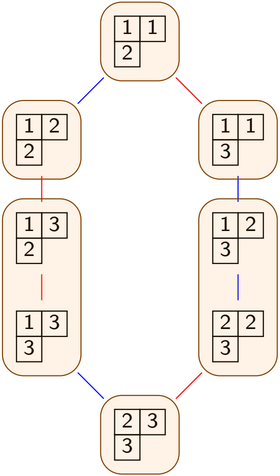
The key polynomial κ_α is the sum of all monomials corresponding to SSYTs with right key at most key(α).

$$\kappa_\alpha = \sum_{\substack{\text{SSYT } T \\ K_+(T) \leq \text{key}(\alpha)}} x^T.$$

Note: "$\leq$" on tableaux is entry-wise comparison.

Comparing κ_α and A_α again





Product of Key Polynomials in Atom Basis

$$\kappa_\alpha \cdot \kappa_\beta = \sum_\gamma d^\gamma_{\alpha,\beta} \cdot A_\gamma.$$

Research Question: Find a combinatorial description of the integer coefficients $d^\gamma_{\alpha,\beta}$.

Conjecture: The coefficients $d^\gamma_{\alpha,\beta}$ are nonnegative integers.

Thank you!

Product of Tableaux

There's a way to define the product $T \cdot U$ of tableaux.

Example:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array}.$$

Applying the definition

Given two compositions α and β,

$$\begin{aligned}
\kappa_\alpha \kappa_\beta &= \left(\sum_{\substack{\text{SSYT } T \\ \mathsf{K}_+(T) \leq \text{key}(\alpha)}} x^T \right) \left(\sum_{\substack{\text{SSYT } U \\ \mathsf{K}_+(U) \leq \text{key}(\beta)}} x^U \right) \\
&= \sum_{\substack{\text{SSYT } T, U \\ \mathsf{K}_+(T) \leq \text{key}(\alpha) \\ \mathsf{K}_+(U) \leq \text{key}(\beta)}} x^T x^U \\
&= \sum_{\substack{\text{SSYT } T, U \\ \mathsf{K}_+(T) \leq \text{key}(\alpha) \\ \mathsf{K}_+(U) \leq \text{key}(\beta)}} x^{T \cdot U}.
\end{aligned}$$

Wishful thinking

It suffices to show that the multiset

$$\left\{ T \cdot U \ :\ \substack{\text{SSYT } T, U \\ \mathsf{K}_+(T) \leq \mathsf{key}(\alpha) \\ \mathsf{K}_+(U) \leq \mathsf{key}(\beta)} \right\}$$

can be partitioned into sets of the form

$$\left\{ V \ :\ \substack{\text{SSYT } V \\ \mathsf{K}_+(V) = \mathsf{key}(\gamma)} \right\}.$$

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can be partitioned into sets of the form

$$\left\{ V \ : \ \substack{\text{SSYT } V \\ \mathsf{K}_+(V) = \text{key}(\gamma)} \right\}.$$

Spoiler: It can't. There are counterexample.

Underlying issue: The structure of tableaux is more strict than the structure of the polynomials/monomials.