Haverford Problem Solving Group
September 21, 2023
Combinatorics

Welcome to the Problem Solving Group, a.k.a. PSG! All materials will be posted to www.guilhermezeus.com/psg. Scan the QR code to join our mailing list.

C1. How many words are at most three letters long and contain only the letters A, B, C, D and E? Here, 'word' refers to any string of letters.

C2. How many words of 7 letters are there with exactly two As, exactly two Bs, and exactly three Cs? Here, 'word' refers to any string of letters.

C3. Emily's broken clock runs backwards at five times the speed of a regular clock. Right now, it is displaying the wrong time. How many times will it display the correct time in the next 24 hours? It is an analog clock (i.e. a clock with hands), so it only displays the numerical time, not AM or PM. Emily's clock also does not tick, but rather updates continuously.

C4. How many ways are there to arrange the numbers 1, 2, 3, 4, 5, 6 on the vertices of a regular hexagon such that exactly 3 of the numbers are larger than both of their neighbors? Rotations and reflections are considered the same.

C5. Nine fair coins are flipped independently and placed in the cells of a 3 by 3 square grid. Let p be the probability that no row has all its coins showing heads and no column has all its coins showing tails.

For problems **C6** and **C7**, solve the problem for $n = 1$, then for $n = 2, 3, 4, 5, 6, 7, \ldots$, until you can spot a pattern. Once you find a pattern, try to explain it.

C6. There are n boxes around a circle, each of which has exactly one chip. In one move, you may simultaneously move any two chips by one place in *opposite* directions. The goal is to get all chips into one box. When can this goal be reached? [Eng98, p. 10]

C7. There are n people standing in a circle. The first person starts with a token. At each turn, the player with the token removes the next person (in clockwise order) from the game, and passes the token to the new next person (in clockwise order). This process goes on until only one person is left. The person left is announced to be the winner. In what position do you want to start the game in order to win? [Erm16]

For more problems, turn the page.

C8. There are 200 students in PSG, organized into 15 groups of varying sizes. Halfway through the semester, they are reoganized into 14 groups (again of varying sizes). Prove that there must be a person whose group size is larger in the second half of the semester. [Loh21]

C9. **(a)** The unit cells of a 5×5 board are painted with 5 colors in a way that every cell is painted by exactly one color and each color is used in 5 cells. Show that exists at least one line or one column of the board in which at least 3 colors were used.

(b) The unit cells of a $n \times n$ board are painted with n colors in a way that every cell is painted by exactly one color and each color is used in n cells. Show that exists at least one line or one column of the board in which at least $\lceil \sqrt{n} \rceil$ colors were used. [AoPS]

C10. There are $2n$ points on the plane, no three collinear. Exactly n of these points are farms, $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$. The remaining n points are wells, $\mathcal{W} = \{W_1, W_2, \dots, W_n\}$. It is intended to build a straight line road from each farm to one well. Show that the wells can be assigned bijectively to the farms, so that none of the roads intersect. [Eng98, p. 42]

C11. Suppose that $2n$ checkers have been placed on $n \times n$ board. Prove that, no matter how they have been placed, there exists a sequence of distinct checkers $C_1, C_2, \dots, C_{2t}$ such that C_1 and C_2 are in the same row, C_2 and C_3 are in the same column, C_3 and C_4 are in the same row, C_4 and C_5 are in the same column, $\dots C_{2t-1}$ and C_{2t} are in the same column. (Note that t can be less than n, that is, it is not required to use all of the checkers in the sequence.) [Loh21]

C12. In 2023, Haverford entered 200 students in the Putnam. It turned out that every one of the 6 questions in the first half was fully solved by at least 120 Haverford students. Show that there must then have been two Haverford students so that each of the 6 problems was solved by at least one of them. [Loh21]

C13. Let D be a directed graph, and let χ be the chromatic number of its underlying graph. Show that D has a directed path of at least χ vertices. [Loh21]

References

[AoPS] *Grid*. Art of Problem Solving Community Forum. URL: https://aops.com/community/c6h620976p7968380.

[Eng98] Arthur Engel. *Problem-Solving Strategies*. Problem Books in Mathematics. New York: Springer, 1998.

[Erm16] Daniel Erman. *The Josephus Problem*. Channel Numberphile. YouTube video. 2016. URL: https://youtu.be/uCsD3ZGzMgE.

[Loh21] Po-Shen Loh. *Combinatorics*. Carnegie Mellon University Putnam Seminar. 2021. URL: https://math.cmu.edu/~ploh/docs/math/2021-295/10-combinatorics.pdf.