Haverford Problem Solving Group
September 28, 2023
Polynomials and Quadratic Functions

Welcome to the Problem Solving Group, a.k.a. PSG! All materials will be posted to www.guilhermezeus.com/psg. Scan the QR code to join our mailing list.

P1. Let the roots of $x^2 - 17x + 13 = 0$ be r and s. What are the values of (a) r^2s^2, (b) $r^2 + s^2$, (c) $r^2s + s^2r$, (d) $r^3 + s^3$.

P2. Find all **real** roots of the polynomials

(a) $p(x) = x^4 - 2x^3 - 2x^2 + 3x + 2$,

(b) $q(x) = x^4 - 4x^3 + 6x^2 - 4x + 6$,

(c) $q(x) = x^4 - 4x^3 + 7x^2 - 2x + 1$.

P3. Let p, q be real numbers, and let a, b, c be distinct real numbers such that $a^3 + pa + q = 0$, $b^3 + pb + q = 0$, and $c^3 + pc + q = 0$. Determine $a + b + c$.

P4. Factor $a^3 + b^3 + c^3 - 3abc$.

P5. Let a, b, c, d be distinct real numbers such that a and b are the roots of the equation $x^2 - 3cx - 8d = 0$, and c and d are the roots of the equation $x^2 - 3ax - 8b = 0$. Determine the sum $a + b + c + d$. [Shi18]

P6. Let a, b, c be real numbers such that the equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have exactly one common root, and such that the equations $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ also have exactly one common root. Determine the sum $a + b + c$. [Shi18]

P7. Let a, b, r, s be real numbers such that the roots of the equation $x^2 - ax + b = 0$ are $\frac{1}{r}$ and $\frac{1}{s}$, and the roots of $x^2 - rx + s = 0$ are a and b. Given that $a > 0$, determine its value. [Shi18]

P8. Let a, b, c be real numbers. Suppose that $x^3 + ax^2 + bx + c = 0$ has three real roots. Prove that $3b \leq a^2$.

P9. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial such that $P(x)$ and $P(P(P(x)))$ have a common root. Prove that $P(0) \cdot P(1) = 0$. [Shi18]

For more problems, turn the page.

P10. Let p and q be integers. Suppose $x^2 + px + q$ is positive for all **integers** x. Show that $x^2 + px + q = 0$ does not have a **real** solution. [Shi18]

P11. Let $f(x) = x^2 + 2007x + 1$. Show that $\underbrace{f(f(\cdots(f}_{n \text{ times}}(x))\cdots)) = 0$ has at least one real root, for any positive integer n. [Shi18]

P12. Let $f(x) = x^2 - 2$. Show that the roots of the equation $\underbrace{f(f(\cdots(f}_{n \text{ times}}(x))\cdots)) = x$ are real and distinct, for any positive integer n. [Lee18, P3]

P13. A quadratic polynomial $P(x)$ satisfies $-1 \le P(x) \le 1$ for $0 \le x \le 1$. Prove that $P(-\frac{1}{2}) \le 7$.

P14. Let $f(x) = x^2 + ax + b$. Suppose $f(f(x)) = 0$ has four distinct real roots, and that the sum of two of them is -1. Prove that $b \le -\frac{1}{4}$. [Shi18]

P15. Let $x_1, x_2, \ldots, x_n$ be real numbers with zero sum, and with sum of squares equals 1. Prove that there exists two of them, x_i and x_j, such that $x_i x_j \le -\frac{1}{n}$. [Shi18]

P16. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x).$$

Prove that $x - 1$ is a factor of $P(x)$. [Lee18, P5]

P17. Find all polynomials $f(x)$ with real coefficients such that $f(x)f(2x^2) = f(2x^3 + x)$. [Lee18, P7]

P18. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10 with real coefficients. Suppose $P(x) = Q(x)$ has no real roots. Prove that $P(x + 1) = Q(x - 1)$ has a real root. [Shi18]

P19. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that, for all $n \in \mathbb{Z}_{>0}$, the polynomial $p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has exactly n distinct real roots? [Lee18, P20]

P20. Find all polynomials $f(x) \in \mathbb{R}[x]$ such that $f(x^2) = f(x)^2$. [Lee18, P25]

References

[Lee18] Hojoo Lee. *Functions, Polynomials, and Sequences*. 2018. URL: https://cosmogeometer.wordpress.com/problems/.

[Shi18] Carlos Shine. *Funções Quadráticas e Polinômios*. 21st Brazilian Olympic Week, in Portuguese. 2018. URL: https://www.obm.org.br/content/uploads/2018/01/Carlos_Shine-_quadratica_2017.pdf.