Haverford Problem Solving Group
October 26, 2023
Number Theory

Let $m \in \mathbb{Z}_{>0}$ be a positive integer, and let $a, b \in \mathbb{Z}$ be integers. **We say that a is *congruent* to b modulo m, denoted by $a \equiv b \pmod{m}$, if m divides $a - b$.** For example, $7 \equiv 2 \pmod{5}$ because $5 \mid (7-2)$. Equivalently, $a \equiv b \pmod{m}$ if and only if a and b have the same remainder when divided by m.

Let $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:

(i) $a + c \equiv b + d \pmod{m}$,

(ii) $a - c \equiv b - d \pmod{m}$,

(iii) $ac \equiv bd \pmod{m}$.

(iv) $a^k \equiv b^k \pmod{m}$ for all $k \in \mathbb{Z}$.

Although we cannot "divide", *sometimes* we can "cancel".

(i) If $\gcd(m, n) = 1$, then $na \equiv nb \pmod{m} \iff a \equiv b \pmod{m}$.

N1. Compute the remainder of

(a) 4^{1234} when divided by 3,

(b) 20^{100} when divided by 17,

(c) 2^{2002} when divided by 101,

(d) $2^{70} + 3^{70}$ when divided by 13.

N2. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

N3. Prove the properties (i) to (v) above.

N4. Show that, for any positive integer n, the number $n^5 - n$ is divisible by 30.

N5. Show that

(a) $2^{32} + 1$ and $2^4 + 1$ are relatively prime, that is, $\gcd(2^{32} + 1, 2^4 + 1) = 1$.

(b) $2^{15} - 1$ and $2^{10} + 1$ are relatively prime, that is, $\gcd(2^{15} - 1, 2^{10} + 1) = 1$.

N6. Find all non-negative integers x and y such that

(a) $2^x = 3^y - 1$,

(b) $2^x = 3^y + 1$,

(c) $2^x = 3^y - 7$.

N7. Let m, n be positive integers such that

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Show that m is divisible by 1979.

For more problems, turn the page.

N8. Let n be a positive integer. Prove that there are pairwise relatively prime integers $k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_n - 1$ is the product of two consecutive integers.

N9 (Fermat–Euler Theorem)**.** Let n be a positive integer. Let a be an integer relatively prime to n. Let $\phi(n)$ be the number of positive integers k up to n that are relatively prime to n. Then,

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

N10. Let p be a prime. Prove that $(p-1)! \equiv -1 \pmod{p}$.

N11. Let p be a prime. Prove that $x^2 \equiv -1 \pmod{p}$ has a solution if and only if $p = 2$ or $p \equiv 1 \pmod{4}$.

N12. Define the *set of sum of squares* as $\mathcal{S} = \{a^2 + b^2 : a, b \in \mathbb{Z}\}$.

(a) Prove that if $m, n \in \mathcal{S}$, then $mn \in \mathcal{S}$.

(b) Let p be a prime number. Prove that $p \in \mathcal{S}$ if and only if $p = 2$ or $p \equiv 1 \pmod{4}$.

(c) Let p be a prime number in $\mathcal{S}$. Prove that $m \in \mathcal{S}$ if and only if $pm \in \mathcal{S}$.

(d) Prove that $n \in \mathcal{S}$ if and only if every prime $p \equiv 3 \pmod{4}$ divides n an even number of times.

N13. Let m, n be positive integers such that $m^2 + n^2$ is divisible by $mn + 1$. Prove that $(m^2 + n^2)/(mn + 1)$ is a perfect square.

N14. Find all pairs of integers (a, b) such that ab divides $a^2 + b^2 + 1$.

N15. Let a, b be positive integers such that $4ab - 1$ divides $(4a^2 - 1)^2$. Prove that $a = b$.

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