Haverford Problem Solving Group
November 02, 2023
Recursion

R1. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes. [Loh23]

R2. How many sequences of 1s and 3s sum to 16? Examples of such sequences are $1, 3, 3, 3, 3, 3$ and $1, 3, 1, 3, 1, 3, 1, 3$. [Loh23]

R3. A sequence is defined by $a_0 = -1, a_1 = 0$, and

$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$

for all positive integers n. Find a_{100}. [Loh23]

R4. Let $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Find a closed formula for the nth Fibonacci number. Use it to show that the ratio F_{n+1}/F_n of successive Fibonacci numbers approaches $\frac{1+\sqrt{5}}{2}$ (the golden ratio) as $n \to \infty$.

R5. How many ways are there to tile a $2 \times n$ rectangle using 1×2 and 2×2 tiles, if the 1×2 tiles come in three colours but the 2×2 tiles only come in one? The 1×2 tiles may be placed either horizontally or vertically. [Tuf]

R6. A type 1 sequence is a sequence with each term 0 or 1 which does not have $0, 1, 0$ as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have $0, 0, 1, 1$ or $1, 1, 0, 0$ as consecutive terms. Show that there are twice as many type 2 sequences of length $n + 1$ as type 1 sequences of length n. [Loh23]

R7. A computer is programmed to randomly generate a string of six symbols using only the letters `A`, `B`, `C`. What is the probability that the string will not contain three consecutive `A`s? [Loh22]

R8. Prove that the Fibonacci numbers satisfy $F_n^2 + F_{n+1}^2 = F_{2n+1}$. [Loh22]

R9. Let F_n be the Fibonacci sequence. Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} F_{n+1}}.$$

[Loh23]

For more problems, turn the page.

R10. Let $x_0 = 1$, and for each $n \geq 0$, let $x_{n+1} = x_n + \frac{1}{x_n}$. Prove that $x_n \to \infty$. [Loh22]

R11 (Zeckendorf's Theorem)**.** Every positive integer can be uniquely represented as the sum of one or more distinct Fibonacci numbers, where no two are consecutive Fibonacci numbers. [Loh23]

R12. For n a positive integer, define $f_1(n) = n$, and then for each i, let $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75) \bmod 17$. [Loh23]

R13. Let $a_3 = \frac{2+3}{1+6}$, and for each $n \geq 4$, let

$$a_n = \frac{n + a_{n-1}}{1 + n a_{n-1}}.$$

Find a_{1995}. [Loh22]

R14. Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0 's and 1 's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1 's. (Thus for $n = 5, 11101$ is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2, f(4) = 3$.) Prove that $f(n) < 1.7^n$ for all n. [Loh22]

R15. Let x be a real number strictly between 0 and 1 . For each positive integer n, define $f_n(t) = t + \frac{t^2}{n}$, and let

$$a_n = f_n\left(f_n\left(\ldots f_n(x)\right)\ldots\right),$$

where f_n is iterated n times. Determine $\lim_{n\to\infty} a_n$. [Loh22]

References

[Loh22] Po-Shen Loh. *Recursions*. Carnegie Mellon University Putnam Seminar. 2022. URL: https://www.math.cmu.edu/~ploh/docs/math/2022-295/08-recursions.pdf.

[Loh23] Po-Shen Loh. *Recursions*. Carnegie Mellon University Putnam Seminar. 2023. URL: https://www.math.cmu.edu/~ploh/docs/math/2023-295/08-recursions.pdf.

[Tuf] Chris Tuffley. *Recurrence relations*. New Zealand Mathematical Olympiad Committee.
