Haverford Problem Solving Group
November 09, 2023
Calculus

Question 0. Are you registered for the Putnam? If not, ask any PSG co-head for help.

K1. Compute the limit

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

[Loh23]

K2. Determine $f'(x)$, if $f(x)=\left[\int_0^{x^2} e^{-t^2}dt\right]^2$.

K3. Find all real functions f for which $\int_0^x f(t)dt=\frac{1}{2}xf(x)$.

K4. Determine

$$\lim_{n\to\infty}\sum_{i=1}^{n^2}\frac{n}{n^2+i^2}.$$

K5. Prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}=\frac{22}{7}-\pi.$$

K6. Let $x_1=\sqrt{5}$ and $x_{n+1}=x_n^2-2$. Compute

$$\lim_{n\to\infty}\frac{x_1x_2x_3\cdots x_{n-1}}{x_n}.$$

[IMC 2010]

K7. Let C be the unit circle $x^2+y^2=1$. A point P is chosen randomly on the circumference C and another point Q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal PQ. What is the probability that no point of R lies outside of C ?

K8. Three infinitely long circular cylinders, each with unit radius, have their axes along the x, y and z axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2+z^2\le 1\}, \{z^2+x^2\le 1\}$, and $\{x^2+y^2\le 1\}$.) [Loh23]

For more problems, turn the page.

K9. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)dx \right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)| \,.$$

K10. Find the volume of the region of points (x,y,z) such that

$$\left(x^2 + y^2 + z^2 + 8\right)^2 \le 36 \left(x^2 + y^2\right).$$

K11. Let P be a convex polygon, let Q be the interior of P, and let $S = P \cup Q$. Let p be the perimeter of P and let A be its area. Given any point (x,y), let $d(x,y)$ be the distance from (x,y) to the nearest point of S. Find constants α, β, and γ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-d(x,y)} dx dy = \alpha + \beta p + \gamma A.$$

K12. Let G_n be the geometric mean of $\binom{n}{0}$, $\binom{n}{1}$, $\ldots$, $\binom{n}{n}$. Compute

$$\lim_{n \to \infty} \sqrt[n]{G_n}.$$

K13. Let $f(x) = \int_0^x \sin\left(t^2 - t + x\right) dt$. Compute $f''(x) + f(x)$, and deduce that $f^{(12)}(0) + f^{(10)}(0) = 0$. (Here, $f^{(10)}$ indicates the 10th derivative.)

K14. Evaluate

$$\int_1^4 \frac{x-2}{\left(x^2+4\right)\sqrt{x}} dx$$

K15. Evaluate

$$\int_1^2 \frac{\ln x}{2 - 2x + x^2} dx$$

K16. Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$$

where $0 \le \arctan(x) < \frac{\pi}{2}$ for $0 \le x < \infty$.

Welcome to the Problem Solving Group, a.k.a. PSG! All materials will be posted to www.guilhermezeus.com/psg. Scan the QR code to join our mailing list.

References

[Loh23] Po-Shen Loh. *Calculus*. Carnegie Mellon University Putnam Seminar. 2023. URL: https://www.math.cmu.edu/~ploh/docs/math/2023-295/04-calculus.pdf.