Haverford Problem Solving Group
November 16, 2023
Problems from the Putnam Competition

Q0. Are you registered for the Putnam‽ If not, ask any PSG co-head for help.

Q1 (Putnam 2017/A1)**.** Let S be the smallest set of positive integers such that
(a) 2 is in S,
(b) n is in S whenever n^2 is in S, and
(c) $(n+5)^2$ is in S whenever n is in S.
Which positive integers are not in S?

Q2 (Putnam 2015/A2)**.** Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

Q3 (Putnam 2014/A1)**.** Prove that every nonzero coefficient of the Taylor series of $(1-x+x^2)e^x$ about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Q4 (Putnam 2016/B1)**.** Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$, $x_{n+1} = \ln(e^{x_n} - x_n)$. Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.

Q5 (Putnam 2015/B1)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a three times differentiable function such that f has at least five distinct real roots. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real roots.

Q6 (Putnam 2013/A1)**.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Q7 (Putnam 2015/A1)**.** Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Q8 (Putnam 2014/A2)**.** Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $1/\min(i,j)$ for $1 \leq i, j \leq n$. Compute $\det(A)$.

For more problems, turn the page.

Q9 (Putnam 2013/B1). For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Q10 (Putnam 2013/A2). Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \dots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Q11 (Putnam 2014/B1). A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \dots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

Q12 (Putnam 2014/B2). Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Q13 (Putnam 2015/B2). Given a list of the positive integers $1, 2, 3, 4, \dots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: $6, 16, 27, 36, \dots$. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.

Q14 (Putnam 2016/A1). Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j} p(x)\right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

Q15 (Putnam 2016/A2). Given a positive integer n, let $M(n)$ be the largest integer m such that $\binom{m}{n-1} > \binom{m-1}{n}$. Evaluate $\lim_{n\to\infty} M(n)/n$.