Haverford Problem Solving Group
February 1, 2024

Here are some strategies highlighted in [Zeitz]:

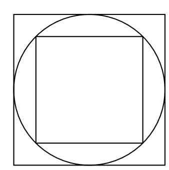
- **orientation**: read the problem carefully. carefully identify the hypothesis and conclusion. think about convenient notation. can you restate the problem in a different way?
- **penultimate step**: what would immediately lead to the conclusion of the problem?
- **get your hands dirty**: experiment with small (and large) cases.
- **wishful thinking**: what would be nice to happen?
- **make it easier**: if the given problem is too hard, solve an easier one.
- **peripheral vision**: don't get locked into one method. try to consciously break or bend the rules.

0. Fill out the introductory survey for the PSG. Your feedback will help us improve the group.

1. Compute $1 + 2 + 3 + \cdots + 100$.

2. Write $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$ as a fraction in lowest terms. [Zeitz, p. 1]

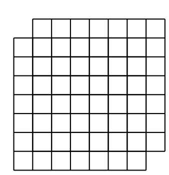
3. A square is inscribed in a circle that is inscribed in a square. Find the ratio of the areas of the two squares. [Zeitz, p. 63]



Challenge: Don't do any algebra.

4. Your cabin is 2 miles due north of a stream that runs east-west. Your grandmother's capin is located 12 miles west and 1 mile north of your cabin. You go from your cabin to your grandmother's, but first visit the stream (to get fresh water). What is the shortest possible length of your trip? [Zeitz, p. 64]

5. Remove two diagonally opposite corners from a 8×8 chessboard. Is it possible to tile this shape with thirty-one 2×1 dominoes? Every square must be covered and no dominoes may overlap. [Zeitz, p. 54]



For more problems, turn the page.

6. Show that the product of four consecutive integers cannot be a perfect square. [Zeitz, p. 4]

7. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible? [Zeitz, p. 34]

8. Lockers in a row are numbered $1, 2, 3, \ldots, 1000$. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Then another person walks by and changes the state of every third locker, starting with locker #3. Then another person walks by and changes the state of every fourth locker, starting with locker #4. This process continues until no more lockers can be altered. Which lockers will be closed? [Zeitz, p. 29]

9. There are 25 people sitting around a circular table, and each person has two cards. One of the numbers $1, 2, 3, \ldots, 25$ is written on each card, and each number is written on exactly two cards. At a signal, each person passes their smaller-numbered card to their right-hand neighbor. Prove that, sooner or later, one of the players will have two cards with the same number. [Zeitz, p. 30]

10. Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

contains the square of an integer. [Zeitz, p. 31]

Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor \sqrt{10} \rfloor = 3$.

References

[Zeitz] Paul Zeitz. *The Art and Craft of Problem Solving*. Wiley, 2006.