Haverford Problem Solving Group
February 8, 2024
Symmetry

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1. Factor $a^3 + b^3 + c^3 - 3abc$. [Zeitz, p. 71]

2. How many of the subsets of the set $\{1, 2, \ldots, 30\}$ have the property that the sum of their elements is greater than 232? [Zeitz, p. 71]

3. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists. [Zeitz, p. 71]

4. Compute $\displaystyle\int_0^{\pi/2} (\cos x)^2 \, dx$. [Zeitz, p. 65]

5. Four bugs are situated at each vertex of a unit square. Suddenly, each bug begins to chase its counterclockwise neighbor. If the bugs travel at 1 unit per minute, how long will it take for the four bugs to crash into one another? [Zeitz, p. 65]

6. Consider the following two-player game. Each player takes turns placing a 2×1 domino on a 10×10 chessboard. The dominoes must be placed so that they do not overlap or hang off the edge of the board. The player who cannot place a domino loses. Who has the winning strategy?

7. A polynomial in several variables is called **symmetric** if it is unchanged when the variables are permuted. For example,

$$f(x, y, z) := x^2 + y^2 + z^2 + xyz$$

is symmetric, since $f(x, y, z) = f(x, z, y) = f(y, x, z) = f(y, z, x) = f(z, x, y) = f(z, y, x)$.

Given three variables x, y, z, we define the **elementary symmetric functions**

$$s_1 = x + y + z, \qquad s_2 = xy + yz + zx, \qquad s_3 = xyz.$$

Elementary symmetric functions can be defined for any number of variables. For example, for four variables x, y, z, w, they are

$$\begin{aligned} s_1 &= x + y + z + w, \\ s_2 &= xy + xz + xw + yz + yw + zw, \\ s_3 &= xyz + xyw + xzw + yzw, \\ s_4 &= xyzw. \end{aligned}$$

For more problems, turn the page.

(a) Verify that

$$\begin{aligned} x^2 + y^2 + z^2 &= (x + y + z)^2 - 2(xy + yz + zx) \\ &= s_1^2 - 2s_2, \end{aligned}$$

where the s_i are the elementary symmetric functions in three variables.

(b) Likewise, express $x^3 + y^3 + z^3$ as a polynomial in the elementary symmetric functions.

(c) Do the same for $(x + y)(x + z)(y + z)$.

(d) Do the same for $xy^4 + yz^4 + zx^4 + xz^4 + yx^4 + zy^4$.

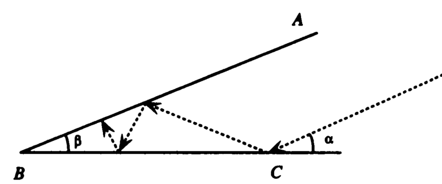
(e) Can any symmetric polynomial in three variables be expressed as a polynomial in the elementary symmetric functions?

(f) Can any polynomial (not necessarily symmetric) in three variables be expressed as a polynomial in the elementary symmetric functions?

(g) Generalize to more variables. If you are confused, look at the two-variable case.

[Zeitz, p. 72]

8. A billiard ball (of infinitesimal diameter) strikes ray $\overrightarrow{BC}$ at point C, with angle of incidence α as shown. The billiard ball continues its path, bouncing off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule "angle of incidence equals angle of reflection." If $AB = BC$, determine the number of times the ball will bounce off the two line segments (including the first bounce, at C). Your answer will be a function of α and β. [Zeitz, p. 72]



9. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

[Zeitz, p. 73]

10. (a) Let K, L, M, N designate the centers of the squares erected on the four sides (outside) of a rhombus. Prove that the polygon $KLMN$ is a square.

(b) Sharpen the problem above by showing that the conclusion still holds if the rhombus is merely an arbitrary parallelogram. [Zeitz, p. 73]

References

[Zeitz] Paul Zeitz. *The Art and Craft of Problem Solving*. Wiley, 2006.