HAVERFORD PROBLEM SOLVING GROUP
FEBRUARY 15, 2024
ALGORITHMS

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0. Sign up to join other members of the PSG at the TRICERATOPS competition, on Saturday, February 24th, 9am–2pm, at Swarthmore College.

An **algorithm** is a set of systematic procedures that solve a problem. By definition, an algorithm must have two properties:

- Solve the problem.
- Terminate.

The second property might seem trivial, but sometimes it is not trivial to show that an algorithm terminates.

Algorithms can be useful when:

- An operation is given in the problem, and we want to organize a sequence of operations to reach a goal.
- You want to build an example.
- We have a complicated and want to reduce it to something simpler.
- You want to represent a set (typically of integers) in some way.
- You want to pack things or separate them into groups.
- Games with winning strategies.

Some techniques are:

- You can use induction to prove that an algorithm works.
- A way to demonstrate that an algorithm terminates is to find a monovariant, that is, something that always decreases or always increases.
- Sometimes an invariant can help in both tasks.
- A simple and effective algorithm in many situations is the greedy algorithm, where we always choose the best option locally. However, the greedy algorithm is not always the best. Be careful!
- Test (or build!) your algorithm with small cases.

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1. There are stones with a total mass of 9 tons. Trucks, each with a capacity of 3 tons, are available to carry these stones. It is only known that no stone weighs more than 1 ton. Regardless of the number of stones and their individual weights, what is the minimum number of trucks required to carry all the stones? Each truck makes only one trip.

2. *Egyptian fractions.* Prove that every positive rational number less than 1 can be written as the sum of fractions with numerator 1 and all distinct denominators.

3. In each cell of a 2000×2000 grid, one of three numbers must be written: -1, 0, or 1. Subsequently, the numbers written in each row and each column are summed, and hence 4000 values are obtained. Show that it is possible to fill the board in such a way that all 4000 values obtained are distinct.

For more problems, turn the page.

4. Determine if there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers that satisfies the following conditions:

- All positive integers appear exactly once in the sequence $a_1, a_2, a_3, \ldots$.
- All positive integers appear exactly once in the sequence $|a_1 - a_2|, |a_2 - a_3|, \ldots$.

5. Scrooge McDuck has three bank accounts, each with an integer amount of money. He only transfers money from one account to another if the balance of the second account is doubled. Prove that Scrooge McDuck can leave all his wealth in two accounts.

6. We have n objects that appear identical, but have different masses, and a two-pan balance that can compare exactly two objects. How many weighings are necessary to determine the heaviest and the lightest object?

7. Several positive integers are written in a row. Alice chooses two neighboring numbers x and y such that $x > y$ and x is to the left of y, we replace (x, y) with $(y + 1, x)$ or $(x - 1, x)$. Prove that Alice can only perform this operation a finite number of times.

8. For each positive integer n, the Haverford Bank issues coins of value $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily distinct values) with a total value of at most $99 + \frac{1}{2}$, prove that it is possible to divide this collection into 100 or fewer groups of coins, each with a total value of at most 1.

9. Let n be a positive integer. Find the smallest integer k with the following property: Given any reals $a_1, \ldots, a_d$ with $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i = 1, 2, \ldots, d$, it is possible to divide these numbers into k or fewer groups such that the sum of the numbers in each group is at most 1.

10. A finite number of squares have a total area of 4. Prove that it is possible to cover a square of side 1 with these squares. Note that the squares can overlap and can partially extend beyond the square.

11. A finite number of rectangles have a total area of 3, and all sides are less than or equal to 1. Demonstrate that with these rectangles it is possible to cover a square of side 1 in such a way that the sides of the rectangles are parallel to the sides of the square. Note that the rectangles can overlap and can partially extend beyond the square.

References

[Shine] Carlos Shine. *Algoritmos.* Treinamento Cone Sul, Etapa, in Brazilian Portuguese. 2019.