Haverford Problem Solving Group
February 29, 2024
Linear Algebra for Combinatorics

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1. Let $A_1, \ldots, A_r$ be distinct subsets of $nu]$ such that $|A_i|$ is odd for all i and $|A_i \cap A_j|$ is even for all $i \neq j$. Find the largest possible value of r.

2. The students of a school go out for ice cream in groups of at least two. After $k > 1$ groups have gone out, each pair of students has gone out together exactly once. Prove that the number of students in the school is at most k. (St. Petersburg)

3. Let $A_1, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ and $k < n$ a positive integer such that $|A_i \cap A_j| = \lambda$ for all $i \neq j$. Prove that $m \leq n$. (Fisher's Inequality)

4. In a party with $2n$ people, each person has an even number of friends. Prove that there are two people who have an even number of friends in common.

5. Let n be an even positive integer and let $S_1, \ldots, S_n$ be subsets of even size of $\{1, 2, \ldots, n\}$. Prove that there exist $i \neq j$ such that $|S_i \cap S_j|$ is even.

6. Let n be a positive integer. Given are $2n+1$ real numbers with the property that, whenever one of them is removed, the remaining $2n$ can be divided into two sets of n elements each having the same sum of elements. Prove that all the numbers are equal. (Classic)

7. Consider a 6×6 board. Each square of the board is painted black or white. It is allowed to choose any $t \times t$ square, $2 \leq t \leq 6$, and invert all the colors of the square. You can do this as many times as you want. Is it always possible to make the entire board black? (Belarus)

8. Let $A_1, A_2, \ldots, A_{n+1}$ be non-empty subsets of $\{1, 2, \ldots, n\}$. Prove that there exist disjoint subsets $I, J \subset [n+1]$ such that

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

(China West 2002)

9. In a competition with n questions taken by m competitors, each question awards a distinct positive amount of points. After the tests were evaluated, it was noticed that it was possible to choose the scoring for each question in such a way that any ranking of the participants could be achieved. What is the largest possible value of m? (Russia 2001)

For more problems, turn the page.

10. Let G be a finite simple graph. There is a lamp at each vertex, and initially, all are turned off. At each step, we can choose a vertex and change the state of the lamps of that vertex and its neighbors. Prove that it is possible to turn all the lamps on simultaneously. (Germany TST 2004)

11. At a mathematical convention, some pairs of mathematicians are friends. At dinner, each participant sits in one of two rooms. Each mathematician insists on having an even number of friends in the same room. Prove that the number of ways to separate the mathematicians into two rooms is a power of 2. (USAMO 2008)

12. Let A be a collection of vectors of length n over $\mathbb{Z}/3\mathbb{Z}$ with the property that for any two distinct vectors $a, b \in A$, there exists some coordinate i such that $b_i \equiv a_i + 1 \pmod 3$. Prove that $|A| \leq 2^n$. (Iran 2006, Sperner Capacity of the Cyclic Triangle)

Hint 1. To find the largest possible value, show an example that works and prove that it is impossible for a larger one. Define the indicator vector $v_i \in \mathbb{F}_2^n$. Prove that $\{v_i\}$ is linearly independent.

Hint 2. Define the $k \times n$ incidence matrix A. Show that $A^T A$ is invertible.

Hint 3. Define the $n \times r$ matrix A of column indicator vectors. Calculate $A^T A$. Show that $A^T A$ is invertible.

Hint 4. Proof by contradiction. Define the $2n \times 2n$ adjacency matrix A. Calculate $(AA)\mathbf{1} = A(A\mathbf{1})$ in two ways.

Hint 5. Proof by contradiction. Define the $n \times n$ matrix A of column indicator vectors. Show that $A^T x = \mathbf{0}$ has a nontrivial solution, hence A is not invertible. Show that $A^T A$ is invertible.

Hint 6. Model the problem in $\mathbb{F}_2^{36}$. Show that the dimension of the generated space is ≤ 35.

Hint 7 (1st solution). Construct an appropriate matrix M with zeros on the diagonal and entries ± 1. Show that the determinant of this matrix is not 0. To do that, show the determinant is an odd integer. Since the matrix is invertible, the solution is unique.

Hint 7 (2nd solution). Solve for positive integers. Solve for integers. Solve for rationals. Solve for reals, using the basis of $\mathbb{R}$ over $\mathbb{Q}$.

Hint 8. Find the linear combination $\sum \lambda_i v_i = \mathbf{0} \in \mathbb{R}^n$. Define $I = \{i : \lambda_i > 0\}$ and $J = \{i : \lambda_i < 0\}$.

Hint 9. Find the linear combination $\sum \lambda_i v_i = \mathbf{0} \in \mathbb{R}^n$. Define $I = \{i : \lambda_i > 0\}$ and $J = \{i : \lambda_i < 0\}$. Show that it is impossible for all from I to rank above all from J.

Hint 10. Define the adjacency matrix A, with entries in $\mathbb{F}_2$ and 1 on the diagonal. Prove that, since A is symmetric, $\operatorname{Im} A = (\operatorname{Ker} A)^\perp$. Prove that $\mathbf{1} \in (\operatorname{Ker} A)^\perp$. Therefore, $\mathbf{1} \in \operatorname{Im} A$.

Hint 11. Define the adjacency matrix A, with entries in $\mathbb{F}_2$ and 1 on the diagonal. Define d as the degree vector of the vertices. Note that a vector $v \in \mathbb{F}_2^n$ is a solution if and only if $(A + \operatorname{diag} d)v = d$. Let $B = A + \operatorname{diag} d$. Show that $d \in \operatorname{Im} B = (\operatorname{Ker} B)^\perp$. The number of solutions will be nonzero and therefore a power of 2, due to the corresponding field $\mathbb{F}_2$.

Solution 12. For an element $a = (a_1, \ldots, a_n) \in A$, consider the linear map $f_a : \mathbb{F}_3^n \to \mathbb{F}_3$ given by $f_a(x) = \prod_{i=1}^n (a_i + 1 - x_i)$. Note that we have $f_a(a) = 1$ and $f_a(b) = 0$ for any elements $a \neq b$ in A. Just as in the first problem, this means that the linear maps $\{f_a : a \in A\}$ are linearly independent. But all of them are linear combinations of monomials $\prod_{i \in S} x_i$ for $S \subseteq [n]$. Since there are 2^n such monomials, we must have $|A| \leq 2^n$.