Haverford Problem Solving Group
March 21, 2024
Hungarian Problems

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1. Prove that every selection of five one-digit positive integers contains a few numbers whose sum is divisible by 10. [KöMaL/C1625]

2. Prove that if a, b, c are real numbers, such that $a+b+c>0, ab+bc+ca>0$ and $abc>0$, then $a>0, b>0$ and $c>0$. [KöMaL/C1627]

3. Find two distinct positive integers n for which $4^n+4^9+4^{100}$ is a perfect square. [KöMaL/C1628]

4. Is it possible that $x, \dfrac{14x+5}{9}$ and $\dfrac{17x-5}{12}$ are all integers? [KöMaL/B5118]

5. Let a and b be real numbers such that $a+b=1$ and $a^2+b^2=2$. Find the value of a^8+b^8. [KöMaL/B5111]

6. A deck of card consists of p red cards and k blue cards. In how many different ways is it possible to select some of the cards so that the number of red cards should be n more than the number of blue cards? [KöMaL/B5112]

7. Let a, b and c denote some given, pairwise relatively prime positive integers. Prove that the equation

$$x^a+y^b=z^c$$

has infinitely many solutions (x, y, z) where x, y and z are positive integers. [KöMaL/B5113]

8. Ali has n coins in his purse, and Baba has $n-1$ purses, initially all empty. Baba is playing the following game: he divides the coins (all in the same purse at start) into two purses, with a_1 coins in one of them and b_1 coins in the other $(a_1, b_1>0)$, and then he writes the product a_1b_1 on a blackboard. Then he continues in the same way: in the k th move $(k=2,3,\ldots)$ he selects a purse containing at least two coins, divides them between two empty purses, with a_k coins in one of them and b_k in the other $(a_k, b_k>0)$, and writes the product a_kb_k on the board.

The game terminates when there is 1 coin in each purse. Then Ali gives as many coins to Baba as the sum of all the products a_kb_k on the board.

What is the maximum number of coins that Baba may get? [KöMaL/B5115]

For more problems, turn the page.

9. Let $a, b, c > 0$ and $x, y, z \geq 0$. Prove that if $x + aby \leq a(y+z), y + bcz \leq b(z+x)$, and $z + cax \leq c(x+y)$, then $x = y = z = 0$ or $a = b = c = 1$. [KöMaL/B5116]

10. The positive integers are coloured in the following manner: the colour of $a+b$ is always uniquely determined by the colours of a and b; that is, if the colour of a and a' is the same, and the colour of b and b' is the same, then $a+b$ and $a'+b'$ also have the same colour. Prove that if there is a colour that is used more than once then the colouring becomes periodic from some number onwards. [KöMaL/B5120]

11. Solve the following simultaneous equations, where $x_1, x_2, \ldots, x_n$ are positive real numbers, and n is a positive integer:

$$x_1 + x_2 + \ldots + x_n = 9,$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = 1.$$

[KöMaL/B5121]

12. ErWin Layup is the best penalty taker of all times in the basketball league of Nowhereland. Although he missed the very first penalty throw of his career, altogether he has only missed 2020 out of his total of 222222 throws.

Statisticians in Nowhereland consider a basketball penalty throw interesting if the ratio of successful penalty throws to all penalty throws, calculated immediately after the throw and expressed as a percentage, is a positive integer. (For example, if a player scores 12 out of a total of 40 throws then his last throw is interesting, since $\frac{12}{40} \cdot 100 = 30 \in \mathbb{N}^+$, while the following throw, which is the 41 st, cannot be interesting, whether successful or not.)

What is the minimum number of interesting penalty throws that ErWin Layup may have had?

[KöMaL/B5122]

13. We colored the n^2 unit squares of an $n \times n$ square lattice such that in each 2×2 part at least two of the four unit squares has the same color. What is the largest number of colors we could have used? [KöMaL/A780]

14. Prove that the edges of a simple planar graph can always be oriented such that the outdegree of all vertices is at most three. [KöMaL/A782]

15. For which n can we partition a regular n-gon into finitely many triangles such that no two triangles share a side? [KöMaL/A710]