Haverford Problem Solving Group
April 18, 2024
Miscellaneous Problems

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1. Evaluate the product
$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$
[IMC 2019/1]

2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$. [IMC 2023/1]

3. Alice writes the matrix $\left(\begin{smallmatrix} 2 & 3 \\ 2 & 4 \end{smallmatrix}\right)$ on the board. Then, Alice performs the following operation on the matrix several times:

- Alice chooses a row or column of the matrix, and
- Alice multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Alice end up with the matrix $\left(\begin{smallmatrix} 2 & 4 \\ 2 & 3 \end{smallmatrix}\right)$ after finitely many steps? [IMC 2023/6]

4. Let $f\colon [0,1] \to (0,\infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0,1]$. Prove that $\int_0^1 f(x)dx \geq 1$. [IMC 2022/1]

5. We color all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same color. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there? [IMC 2022/5]

6. Let A be a real $n \times n$ matrix such that $A^3 = 0$. Prove that there is unique real $n \times n$ matrix X that satisfies the equation
$$X + AX + XA^2 = A,$$
and express the unique solution in terms of A. [IMC 2021/1]

7. Find all twice continuously differentiable functions $f\colon \mathbb{R} \to (0,\infty)$ satisfying
$$f''(x)f(x) \geq 2f'(x)^2$$
for all $x \in \mathbb{R}$. [IMC 2020/5]

For more problems, turn the page.

8. Let n be a positive integer. Compute the number of words w that satisfy the following three properties:

- w consists of n letters from the alphabet $\{a, b, c, d\}$.
- w contains an even number of a's.
- w contains an even number of b's.

For example, for $n = 2$, there are 6 such words: aa, bb, cc, dd, cd, dc. [IMC 2020/1]

9. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $(f(0) - g'(0))(g'(1) - f(1)) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$. [IMC 2019/6]

10. Let k be a positive integer. Find the smallest positive integer n for which there exists k nonzero vectors $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal. [IMC 2018/6]

11. Prove that

$$\sum_{k=0}^{n} \frac{2^{2^k} \cdot 2^{k+1}}{2^{2^k} + 3^{2^k}} < 4$$

holds for all positive integers n. [KöMaL, A854]

12. In scalene triangle ABC the shortest side is BC. Let points M and N be chosen on sides AB and AC, respectively, such that $BM = CN = BC$. Let I and O denote the incenter and circumcenter of triangle ABC, and let D and E denote the incenter and circumcenter of triangle AMN. Prove that lines IO and DE intersect each other on the circumcircle of triangle ABC. [KöMaL, A855]

13. In a rock-paper-scissors round-robin tournament any two contestants play against each other ten times in a row. Each contestant has a favorite strategy, which is a fixed sequence of ten hands (for example, RRSPPRSPPS), which they play against all other contestants. At the end of the tournament it turned out that every player won at least one hand (out of the ten) against any other player.

Prove that at most 1024 contestants participated in the tournament. [KöMaL, A856]